SPECTRAL PROBLEMS IN ELASTICITY. SINGULAR BOUNDARY PERTURBATIONS

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ABSTRACT. The three-dimensional spectral elasticity problem is studied in an anisotropic and inhomogeneous solid with small defects, i.e., inclusions, voids, and microcracks. Asymptotics of eigenfrequencies and the corresponding elastic eigenmodes are constructed and justified. New technicalities of the asymptotic analysis are related to variable coefficients of differential operators, vectorial setting of the problem, and usage of intrinsic integral characteristics of defects. The asymptotic formulae are developed in a form convenient for application in shape optimization and inverse problems.

Keywords: Singular perturbations; Spectral problem; Asymptotics of eigenfunctions and eignevalues; Elasticity boundary value problem
MSC: Primary 35C20, 35J25, 35B40; Secondary 35J20, 46E35,49Q10, 74P15

1. INTRODUCTION.

1.1. **Shape optimisation problems for eigenvalues.** In the paper asymptotic analysis of eigenvalues and eigenfunctions is performed with respect to singular perturbations of geometrical domains (see Fig. 1).

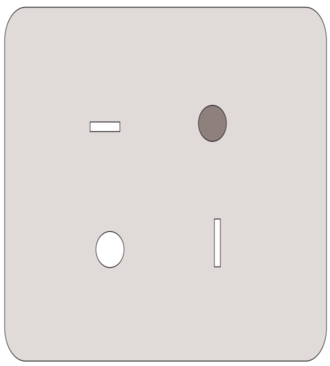

Fig. 1

The case of low frequencies is considered for elasticity spectral problems in three spatial dimensions. The results established here can be directly used in some applications, for example in inverse problems of identification of small defects in the body based on the observation of elastic eigenmodes. Compared to the existing results in the literature, the technical difficulties of the present paper mainly concern vectorial setting of boundary

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value problems, anisotropy of physical properties, and variable coefficients of differential operators, i.e., inhomogeneity of elastic materials. Exisiting results on elasticity problems with singular perturbations of boundaries (see monographs [38, 41] and [18]) deal with homogeneous, mainly isotropic elastic bodies. For a system of differential equations, an asymptotic analysis is required to be much more elaborated and direct adopting of the methods proper for scalar equations may lead to an unfortunate mistake (cf. [19] and corrections in [1]). The known results are given in particular for singular perturbations of isolated points of the boundary (small holes in the domain, see [16], [17], [5], [1], [18], [37] and others), perturbations of straight boundaries including perturbations by changing the type of boundary conditions (cf. [2]-[3]), and the dependence of the obtained results in more general geometrical domains on the curvature is clarified in [31, 32, 8] in the case of scalar equations. The most of attention is paid in the present paper to derivation of explicit formulae for solutions and extraction of principal characteristics of elastic fields and defects which influence these formulae. To this end, we employ matrix/column notation, use the notion of elastic polarization matrix (tensor), and perform certain additional technical calculations which are not needed in the case of homogeneous, isotropic elastic materials.

Small defects can be regarded as singular perturbations of the interior piece of the boundary of the body. In this way we can consider e.g., the finite number of isolated points which approximate small cavities. More generally, by means of asymptotic analysis we can model the creation of caverns, i.e., some piece of material is taken off from the elastic body. We can also fill the cavern with some other elastic material and model such a phenomenon by formation of one or more inclusions in the body.

Roughly speaking, the influence of a substantial change of local properties of the elastic body cannot be analysed by the classical tools of the shape sensitivity analysis or any other type of sensitivity analysis, but it requires the application of asymptotic methods. Especially, such methods turn out to be of importance for the microcracks, since the microcrack implies the creation of a new portion of internal boundary in the body, which cannot be taken into account in the framework of classical sensitivity analysis based on regular perturbations of the coefficients and of the boundary. The asymptotic methods seem to be the only avalaible tool to perform the efficient analysis of solutions, eigenvalues and eigenfunctions, and of shape functionals, in general setting. The internal perturbations of the domain by creation of small openings or holes, but very close to the boundary (see Fig. 2)

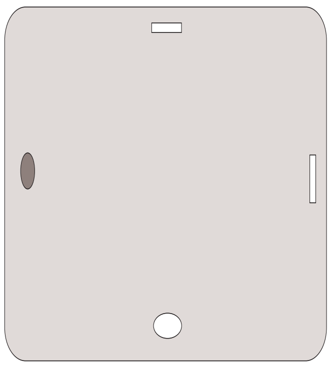

Fig. 2

will be a subject of another paper. Here, we consider small caverns inside the body, i.e., at a distance form the exterior boundary.

We leave aside an important and still not completed topic related to the so-called concentrated masses. Since the pioneering work [39] of E. Sanchez-Palencia, a lot of attention has been paid to mathematical analysis of vibrations of elastic bodies, with small parts wich are very heavy (e.g., pellets in an aspic or in a meat-jelly); see papers [40, 35, 20, 9, 11, 4, 28], as well as the monographs [41, 36] in an incomplete list. Such problems are the best examples of the topping role of the boundary layer effect. Although we analyse the boundary layers in details, the purposes of the present paper is essentially different so that we cannot mutually serve for an analysis of concentrated masses.

1.2. **Preliminaries, anisotropic inhomogeneous elastic body.** Let us consider in three spatial dimensions the elasticity problem for an elastic body Ω, written in the matrix/column notation, see e.g., [10], [24] for more details,

$$\mathcal{D}(-\nabla_x)^\top \mathcal{A}(x)\mathcal{D}(\nabla_x)u \quad = \quad 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$\mathcal{D}(n)^\top \mathcal{A}(x)\mathcal{D}(\nabla_x)u \quad = \quad g^\Omega \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\mathcal{A}$ is a symmetric positive definite matrix function in $\overline{\Omega}$ of size $6{\times}6$, with measurable or smooth elements, consisting of the elastic material moduli (the Hooke's or stiffness matrix) and $\mathcal{D}(\nabla_x)$ is (6×3)-matrix of the first order differential operators,

$$\mathcal{D}(\xi)^\top = \begin{bmatrix} \xi_1 & 0 & 0 & 0 & 2^{-1/2}\xi_3 & 2^{-1/2}\xi_2 \\ 0 & \xi_2 & 0 & 2^{-1/2}\xi_3 & 0 & 2^{-1/2}\xi_1 \\ 0 & 0 & \xi_3 & 2^{-1/2}\xi_2 & 2^{-1/2}\xi_1 & 0 \end{bmatrix} , \tag{3}$$

$u = (u_1, u_2, u_3)^\top$ is displacement column, $n = (n_1, n_2, n_3)^\top$ is the unit outward normal vector on $\partial\Omega$ and $^\top$ stands for transposition. In this notation the strain $\varepsilon(u; x)$ and stress $\sigma(u; x) = \mathcal{A}(x)\mathcal{D}(\nabla_x)u(x)$ columns are given respectively by

$$\mathcal{D}(\nabla_x)u = \varepsilon(u) = \left(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33},\ \sqrt{2}\varepsilon_{23},\ \sqrt{2}\varepsilon_{31},\ \sqrt{2}\varepsilon_{12}\right)^\top , \tag{4}$$

$$\mathcal{A}\mathcal{D}(\nabla_x)u = \sigma(u) = \left(\sigma_{11}, \sigma_{22}, \sigma_{33},\ \sqrt{2}\sigma_{23},\ \sqrt{2}\sigma_{31},\ \sqrt{2}\sigma_{12}\right)^\top . \tag{5}$$

The factors $2^{-1/2}$ and $\sqrt{2}$ imply that the norms of strain and stress tensors coincide with the norms of columns (4) and (5), respectively. From the latter property in the matrix/column notation, any orthogonal transformation of coordinates in $\mathbb{R}^3$ gives rise to orthogonal transformations of columns (4) and (5) in $\mathbb{R}^6$ (cf. [[24];Ch. 2]).

Remark 1.1. *The strains* (4) *and the stresses* (5) *degenerate on the space of rigid motions,*

$$\mathcal{R} = \{d(x)c \ : \ c \in \mathbb{R}^6\} , \quad \dim\mathcal{R} = 6 , \tag{6}$$

where

$$d(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2^{-1/2}x_3 & 2^{-1/2}x_2 \\ 0 & 1 & 0 & 2^{-1/2}x_3 & 0 & -2^{-1/2}x_1 \\ 0 & 0 & 1 & -2^{-1/2}x_2 & 2^{-1/2}x_1 & 0 \end{bmatrix} . \tag{7}$$

This subspace plays a critical role in many questions in the elasticity theory, it appears also in the so-called polynomial property [21, 22] *(see also* [30]*).*

The following equalities can be verified by a direct computation,

$$\mathcal{D}(\nabla_x)\mathcal{D}(x)^\top = \mathbb{I}_6 , \quad \mathcal{D}(\nabla_x)d(x) = \mathbb{O}_6 , \tag{8}$$
$$d(\nabla_x)^\top \mathcal{D}(x)^\top{}_{|x=0} = \mathbb{I}_6 , \quad d(\nabla_x)^\top d(x)_{|x=0} = \mathbb{I}_6 ,$$

where $\mathbb{I}_N$ and $\mathbb{O}_N$ are the unit and null $(N\times N)$-matrices, respectively.

The boundary load g^Ω is supposed to be self equilibrated in order to assure the existence of a solution to the elasticity problem,

$$\int_{\partial\Omega} d(x)^\top g^\Omega(x) ds_x = 0 \in \mathbb{R}^6 \ . \tag{9}$$

2. Vibrations of elastic bodies.

Consider inhomogenuous anisotropic elastic body $\Omega \subset \mathbb{R}^3$ with the Lipschitz boundary $\partial\Omega$. Spectral problems for the body are formulated in a fixed Cartesian coordinate system $x = (x_1, x_2, x_3)^\top$, and in the matrix notation.

We assume that the matrix $\mathcal{A}$ of elastic moduli is a matrix function of the spatial variable $x \in \mathbb{R}^3$, symmetric and positive definite for $x \in \Omega \cup \partial\Omega$. The problem on eigenvibrations of the body Ω takes the form

$$\mathcal{L}(x, \nabla_x)u(x) := \mathcal{D}(-\nabla_x)^\top \mathcal{A}(x)\mathcal{D}(\nabla_x)u(x) = \lambda\gamma(x)u(x) \quad x \in \Omega, \tag{10}$$

$$\mathcal{N}^\Omega(x, \nabla_x)u = \mathcal{D}(n)^\top \mathcal{A}(x)\mathcal{D}(\nabla_x)u(x) = 0, \quad x \in \Sigma, \quad u(x) = 0, \quad x \in \Gamma, \tag{11}$$

where $\gamma > 0$ is the material density, λ is an eigenvalue, the square of eigenfreguency. The part Γ of the surface $\partial\Omega$ is clamped, and the first boundary condition is prescribed on the traction free remaining part $\Sigma = \partial\Omega\setminus\overline{\Gamma}$ of the surface. We denote by $\overset{\circ}{H}{}^1(\Omega;\Gamma)^3$ the energy space, i.e., the subspace of the Sobolev space $H^1(\Omega)^3$ with null traces on the subset Γ. The variational formulation of problem (10)-(11) reads :
Find a non trivial function $u \in \overset{\circ}{H}{}^1(\Omega;\Gamma)^3$ and a number λ such that for all test functions $v \in \overset{\circ}{H}{}^1(\Omega;\Gamma)^3$ the following integral identity is verified

$$(\mathcal{A}\mathcal{D}u, \mathcal{D}v)_\Omega = \lambda(\gamma u, v)_\Omega \ , \tag{12}$$

where $(,)_\Omega$ is the scalar product in the Lebesgue space $L^2(\Omega)$.
If the stiffness matrix $\mathcal{A}$ and the density γ are measurable functions of the spatial variables x, and in addition uniformly positive definite and bounded, then the variational problem (12) admits normal positive egenvalues λ_p, which form the sequence

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_p \le \cdots \to \infty \tag{13}$$

taking into account its multiplicities, and the corresponding eigenfunctions $u_{(p)}$, the elastic vibration modes, are subject to the orthogonality and normalization conditions

$$(\gamma u_{(p)}, u_{(q)})_\Omega = \delta_{p,q} \ , \quad p, q \in \mathbb{N} := \{1, 2, \dots\} \ , \tag{14}$$

where $\delta_{p,q}$ is the Kronecker symbol.

In the sequel it is assumed that elements of the matrix $\mathcal{A}$ and the density γ are smooth functions in Ω, continuous up to the boundary. In such the case Ω is called a *smooth* inhomogenuous body. For such a body the elastic modes $u_{(p)}$ are smooth functions in the interior of Ω, and up to the boundary in the case of the smooth surface $\partial\Omega$. We have also the equivalence between the variational form and the differential form (10)-(11) of the spectral problem. We require only the *interior* regularity of elastic modes in the sequel, in any case the elastic modes have singularities on the collision line $\overline{\Sigma} \cap \overline{\Gamma}$ and therefore, are excluded from the Sobolev space $H^2(\Omega)^3$.

Along with the smooth inhomogenuous body Ω let us consider a body Ω_h with defects; here $h > 0$ stands for a small dimensionless geometrical parameter, which describes the relative size of defects. Actually, we select in the interior of Ω the points $P^1, \dots, P^J$ and denote by $\omega_1, \ldots, \omega_J$ elastic bodies bounded by the Lipschitz surfaces $\partial\omega_1, \ldots, \partial\omega_J$, furthemore, for the sake of simplicity we assume that the origin O belongs to ω_j, $j = 1, \ldots, J$. The body with defects is defined by

$$\Xi(h) = \Omega(h) \cup \omega_1^h \cup \cdots \cup \omega_J^h \tag{15}$$

where

$$\omega_j^h = \left\{ x : \xi^j := h^{-1}(x - P^j) \in \omega_j \right\}, \quad \Omega(h) = \Omega \setminus \bigcup_{j=1}^{J} \overline{\omega_j^h}. \tag{16}$$

The stiffness matrix and the density of the *composite* body (15) take the form

$$\mathcal{A}^h(x) = \begin{cases} \mathcal{A}(x), & x \in \Omega(h); \\ \mathcal{A}_{(j)}(\xi^j), & x \in \omega_j^h; \end{cases} \qquad \gamma^h(x) = \begin{cases} \gamma(x), & x \in \Omega(h); \\ \gamma_j(\xi^j), & x \in \omega_j^h. \end{cases} \tag{17}$$

The matrices $\mathcal{A}$ and $\mathcal{A}_{(j)}$ as well as the scalars γ and $\gamma_{(j)}$ are different from each other, i.e., ω_j^h are inhomogenuous inclusions of small diameters. We assume that $\mathcal{A}_{(j)}$ and $\gamma_{(j)}$ are measurable, bounded and positive definite uniformly on ω_j. In particular, for almost all $\xi \in \omega_j$ the eigenvalues of the matrix $\mathcal{A}_{(j)}(\xi)$ are bounded from below by a constant $c_j > 0$. There is no special assumption on the relation between the properties of the inclusions and of the matrix (body without inclusions), we assume only that the densities γ, $\gamma_{(j)}$, and entries of the matrices $\mathcal{A}$, $\mathcal{A}_{(j)}$ are of similar orders, respectively. We point out that in the framework of our asymptotic analysis, in section 4 there are performed the limit pasages $\mathcal{A}_{(j)} \to 0$ and $\gamma_{(j)} \to 0$ (a hole) as well as $\mathcal{A}_{(j)} \to \infty$ and $\gamma_{(j)} \to \infty$ (an absolutely rigid inclusion). However, the passage $\gamma_{(j)} \to \infty$ with the fixed matrix function $\mathcal{A}_{(j)}$ (heavy concentrated masses) can be analysed with some other ansätze, cf. [38, 40, 4].

In the fracture mechanics, the most intereting case is the weakening of elastic material due to the crack formation. The cracks are modelled by two-sided, two dimensional surfaces, with the first boundary conditions from (11) prescribed on the both crack lips, i.e. the surface is traction free from both sides. The case of a microcrack is not formally included in our problem statement, since we assume that the defect ω_j is of positive volume and with the Lipschitz boundary $\partial\omega_j$. However, the asymptotic procedure works also for the cracks. Small changes which are required in the justification part, are given separately (see the end of section 4, proof of Proposition 5.1 and Remark 5.1). The polarization matrices for the cracks can be found in [43], [33].

The exchange of γ and $\mathcal{A}$ by γ^h and $\mathcal{A}^h$ from (17), respectively, transforms (12) in the integral identity for the body weakened by the defects $\omega_1^h, \ldots, \omega_J^h$, this integral identity is further denoted by $(12)^h$. We observe also, that for smooth stiffness matrix $\mathcal{A}$ and the density γ the differential problem for vibrations of a composite body does not consist only of the system of equations, denoted in our notation by $(10)^h$, restricted to the union of domains (15), along with the boundary conditions $(11)^h$, but in addition it contains transmition conditions on the surface $\partial\omega_j^h$ where the ideal contact is assumed. Since we use only the variational formulations of the spectral problems, the transmission conditions are not explicitely written. In a similar way as for problem (13), there is the sequence of eigenvalues for the problem $(12)^h$

$$0 < \lambda_1^h \le \lambda_2^h \le \cdots \le \lambda_p^h \le \cdots \to +\infty, \tag{18}$$

and the corresponding eigenfunctions $u_{(j)}^h$ meet the orthogonality and normalization conditions

$$(\gamma^h u_{(p)}^h, u_{(q)}^h)_\Omega = \delta_{p,q}, \quad p, q \in \mathbb{N} \tag{19}$$

3. Formal construction of asymptotics

We introduce the following asymptotic ansätze for eigenvalues and eigenfunctions in problem $(12)^h$

$$\lambda_p^h = \lambda_p + h^3\mu_p + \dots, \tag{20}$$

(21)
$$u_{(p)}^h(x) = u_{(p)}(x) + h\sum_{j=1}^{J}\chi_j(x)\left(w_{(p)}^{1j}\left(h^{-1}\left(x-P^j\right)\right) + hw_{(p)}^{2j}\left(h^{-1}\left(x-P^j\right)\right)\right) + h^3 v_{(p)} + \dots$$

where $\chi_j \in C_c^\infty(\Omega)$, $j = 1,\dots,J$, are cut-off functions, with non overlaping supports in Ω, and for each $j, \chi_j(x) = 1$ for $x \in \omega_j$ and $\chi_i(P^j) = \delta_{i,j}$.

First, we assume that the egenvalue $\lambda = \lambda_p$ in problem (12) is simple, and for brevity the subscript p is omitted. The corresponding eigenfunction $u = u_{(p)} \in \overset{\circ}{H}{}^1(\Omega;\Gamma)^3$, normalized by condition (14), is smooth in the interior of the domain Ω.

Columns of the matrices $d(x)$ and $\mathcal{D}(x)^\top$ form a basis in twelve dimensional space of linear vector functions in $\mathbb{R}^3$. In this way, the Taylor formula takes the form

$$u(x) = d(x-P^j)a^j + \mathcal{D}(x-P^j)^\top\varepsilon^j + O(|x-P^j|^2)\ , \tag{22}$$

and, by equalities (4), (5) and (8), the columns

$$a^j = d(\nabla_x)^\top u(P^j)\ , \quad \varepsilon^j = \mathcal{D}(\nabla_x)u(P^j)\ ,$$

represent the column of rigid motions, and of strains, at the point P^j. Since in the vicinity of the inclusion ω_j^h we have

$$\varepsilon(u;x) = \varepsilon^j + O(x) = \varepsilon^j + O(h)\ ,$$

the main terms of discrepancies, left by the field u in the problem $(12)^h$ for the composite body Ω^h, appear in the system of equations in ω_j^h and in the transmition conditions on $\partial\omega_j^h$. For the compensation of the discrepancies are used the special solutions of the elasticity problem in a homogenuous space with the inclusion ω_j of unit size

$$\begin{aligned} &L^{0j}(\nabla_\xi)W^{jk}(\xi) := \mathcal{D}(-\nabla_\xi)^\top\mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)W^{jk}(\xi) = 0, \quad \xi \in \Theta_j = \mathbb{R}^3\setminus\overline{\omega_j},\\ &L^j(\xi,\nabla_\xi)W^{jk}(\xi) := \mathcal{D}(-\nabla_\xi)^\top\mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)W^{jk}(\xi) = \mathcal{D}(\nabla_\xi)\mathcal{A}_{(j)}(\xi)\mathbf{e}_k, \quad \xi \in \omega_j,\\ &W_+^{jk}(\xi) = W_-^{jk}(\xi), \quad \mathcal{D}(\nu(\xi))^\top(\mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)W_-^{jk}(\xi)\\ &\qquad - \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)W_+^{jk}(\xi)) = \mathcal{D}(\nu(\xi))^\top(\mathcal{A}(P^j) - \mathcal{A}_{(j)}(\xi))\mathbf{e}_k, \quad \xi \in \partial\omega_j. \end{aligned} \tag{23}$$

Here ν is the unit vector of the exterior normal on the boundary $\partial\omega_j$ of the body ω_j, $\mathbf{e}_k = (\delta_{1,k},\dots,\delta_{6,k})^\top$ is a orthant in the space $\mathbb{R}^6$, W_+ and W_- are limit values of the function W on the surface $\partial\omega_j$ evaluated from outside and from inside of the inclusion ω_j, respectively.

We denote by Φ^j the fundamental (3×3)-matrix of the operator $L^{0j}(\nabla_\xi)$ in $\mathbb{R}^3$. This matrix is infinitely differentiable in $\mathbb{R}^3\setminus O$ and enjoys the following positive homogeneity property

$$\Phi(t\xi) = t^{-1}\Phi(\xi)\ , \quad t > 0\ . \tag{24}$$

It is known (see, e.g., [[27], Ch. 6]), that the solutions W^{jk} of problem (23) admit the expansion

$$W^{jk}(\xi) = \sum_{p=1}^{6} M_{kp}^j \sum_{q=1}^{3}\mathcal{D}_p^q(\nabla_x)\Phi^{jq}(\xi) + O(|\xi|^{-3}), \quad \xi \in \mathbb{R}^3\setminus\mathbb{B}_R, \tag{25}$$

where $\mathcal{D}_p = (\mathcal{D}_p^1, \mathcal{D}_p^2, \mathcal{D}_p^3)$ is a line of the matrix $\mathcal{D}$ (see (3)), $\Phi^{j1}, \Phi^{j2}, \Phi^{j3}$ are columns of the matrix Φ^j, and the radius R of the ball $\mathbb{B}_R = \{\xi \ : \ |\xi| < R\}$ is chosen such that $\overline{\omega}_j \subset \mathbb{B}_R$. The coefficients M^j_{kp} in (25) form the (6×6)-matrix M^j which is called the *polarization matrix* of the elastic inclusion ω_j (see[43, 23] and also [[27]; Ch. 6], [5], [30]). Some properties of the polarization matrix, and some comments on the solvability of problem (23) are given in section 4.

The columnes $W^{j1}, \dots, W^{j6}$ compose the (3×6)-matrix W^j and we set

$$w^{1j}(\xi) = W^j(\xi)\varepsilon^j \ . \tag{26}$$

In section 5 it is verified, that the right choice of boundary layer is given by formula (26), since it compensates the main terms of discrepancies. From (25) and (26) it follows that

$$w^{1j}(\xi) = (M^j\mathcal{D}(\nabla_\xi)\Phi^j(\xi)^\top)^\top\varepsilon^j + O(|\xi|^{-3}) \ , \quad \xi \in \mathbb{R}^3 \setminus \mathbb{B}_R \ . \tag{27}$$

Relation (27) can be differentiated term by term on the set $\mathbb{R}^3 \setminus \mathbb{B}_R$ under the rule $\nabla_\xi O(|\xi|^{-p}) = O(|\xi|^{-p-1})$ for the remainder.

In view of (24) the detached asymptotics term equals

$$h^2(M^j\mathcal{D}(\nabla_x)\Phi^j(x-P^j)^\top)^\top\varepsilon^j \ . \tag{28}$$

It produces discrepancies of order h^3 (we point out that there is the factor h on w^{1j} in (21)), which should be taken into account when constructing the regular type term $h^3 v$. On the other hand, discrepancies of the same order h^3 are left in the problem for v by the subsequent term $h^2 w(h^{-1}(x - P^j))$, which solves the transmission problem analoguous to (23)

$$L^{0j}(\nabla_\xi)w^{2j}(\xi) = F^{0j}(\xi), \quad \xi \in \Theta_j, \quad L^j(\xi, \nabla_\xi)w^{2j}(\xi) = F^j(\xi), \quad \xi \in \omega_j, \tag{29}$$

$$\begin{aligned} w_+^{2j}(\xi) = w_-^{2j}(\xi); \quad & \mathcal{D}(\nu(\xi))^\top(\mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)w_-^{2j}(\xi) \\ & - \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)w_+^{2j}(\xi)) = G^j(\xi), \quad \xi \in \partial\omega_j, \end{aligned} \tag{30}$$

and with the decay rate $O(|\xi|^{-1})$ at $|\xi| \to \infty$, smaller compared to the decay rate of w^{1j}.

Now, we evaluate the right-hand sides of the problems (29), (30). First, by the representation of the stiffness matrix

$$\mathcal{A}(x) = \mathcal{A}(P^j) + (x - P^j)^\top\nabla_x\mathcal{A}(P^j) + O(|x - P^j|^2) \tag{31}$$

and the corresponding splitting of differential operator with the variable coefficients $\mathcal{L}^0(x, \nabla_x)$ from (10), we find that the right-hand side of system (29) is the main term of the expression

$$-\mathcal{L}^0(x, \nabla_x)w^{1j}(h^{-1}(x-P^j)) = h^{-1}\mathcal{D}(\nabla_\xi)^\top(\xi^\top\nabla_x\mathcal{A}(P^j))\mathcal{D}(\nabla_\xi)w^{1j}(\xi) + \dots =: h^{-1}F^{0j}(\xi) + \dots \tag{32}$$

We note that $L^{0j}(\nabla_x)w^{1j}(h^{-1}(x - P^j)) = 0$ in (32), and the dots $\dots$ stand for the terms of lower order, which are unimportant for our asymptotic analysis. The following discrepancy appears in the second transmission condition (30) :

$$\begin{aligned} G^j(\xi) = \ & \mathcal{D}(\nu(\xi))^\top(\xi^\top\nabla_x\mathcal{A}(P^j))(\mathcal{D}(\nabla_\xi)w^{1j}(\xi) + \varepsilon^j) \\ & + \mathcal{D}(\nu(\xi))^\top(\mathcal{A}(P^j) - \mathcal{A}_{(j)}(\xi))\mathcal{D}(\nabla_\xi)U^j(\xi). \end{aligned} \tag{33}$$

The second term comes out from the elaborated Taylor formula (31)

$$u(x) = d(x - P^j)a^j + \mathcal{D}(x - P^j)^\top\varepsilon^j + U^j(x - P^j) + O(|x - P^j|^3) \tag{34}$$

and involves the quadratic vector function

$$U^j(x-P^j)=\sum_{p,q=1}^{3}(x_p-P^j_p)(x_q-P^j_q)U^{jpq},\quad U^{jpq}=\frac{1}{2}\frac{\partial^2 u}{\partial x_p\partial x_q}(P^j). \tag{35}$$

Finally, the right-hand side of system (29) takes the form

$$F^j(\xi)=-\lambda\gamma_j(\xi)u(P^j)+\mathcal{D}(\nabla_\xi)^\top\mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)U^j(\xi). \tag{36}$$

Besides the term obtained from the quadratic vector function (35) in the Taylor formula (34), the expression (36) contains the discrepancy $\lambda\gamma_j u(P^j)$ which originates from the inertial term $\lambda^h\gamma_j u^h$ in accordance to the asymptotic ansätze (35) and (35).

In order to establish properties of solutions to the problem (29), (30), we need some complementary results.

Lemma 3.1. *Assume that* $Z(\xi)=\mathcal{D}(\nabla_\xi)^\top Y(\xi)$ *and*

$$Y(\xi)=\rho^{-2}\mathfrak{Y}(\theta),\quad Z(\xi)=\rho^{-3}\mathfrak{Z}(\theta), \tag{37}$$

where (ρ,θ) *are spherical coordinates and* $\mathfrak{Y}\in C^\infty(\mathbb{S}^2)^6$, $\mathfrak{Z}\in C^\infty(\mathbb{S}^2)^3$ *are smooth vector functions on the unit sphere. The model problem*

$$L^{0j}(\nabla_\xi)X(\xi)=Z(\xi),\quad \xi\in\mathbb{R}^3\setminus\{0\}, \tag{38}$$

admits a solution $X(\xi)=\rho^{-1}\mathfrak{X}(\theta)$*, which is defined up to the term* $\Phi^j(\xi)c$ *with* $c\in\mathbb{R}^3$*, and becomes unique under the orthogonality condition*

$$\int_{\mathbb{S}^2}\mathcal{D}(\xi)^\top\mathcal{A}^0(P^j)\mathcal{D}(\nabla_\xi)X(\xi)ds_\xi=0\in\mathbb{R}^3. \tag{39}$$

Proof After separating variables and rewriting the operator $L^{0j}(\nabla_\xi)=r^{-2}\mathfrak{L}(\theta,\nabla_\theta,r\partial/\partial r)$ in the spherical coordinates, the system (38) takes the form

$$\mathfrak{L}^j(\theta,\nabla_\theta,-1)\mathfrak{X}(\theta)=\mathfrak{Z}(\theta),\quad \theta\in\mathbb{S}^2. \tag{40}$$

Since $\mathfrak{L}(\theta,\nabla_\theta,0)$ is the formally adjoint operator for $\mathfrak{L}^j(\theta,\nabla_\theta,-1)$ (see, for example, [[27]; Lemma 3.5.9]), the compability condition for the system of differential equations (40) implies the equality

$$\int_{\mathbb{S}^2}\mathfrak{Z}(\theta)ds_\theta=0\in\mathbb{R}^3\ . \tag{41}$$

The equality represents the orthogonality condition in the space $L^2(\mathbb{S}^2)$ of the right-hand side $\mathfrak{Z}$ of system (40) to the solutions of the system

$$\mathfrak{L}^j(\theta,\nabla_\theta,0)\mathfrak{V}(\theta)=0\quad \theta\in\mathbb{S}^2\ , \tag{42}$$

which are nothing but constant columns. Indeed, after transformation to the Cartesian coordinate system ξ equations (42) take the form $L^{0j}(\nabla_\xi)V(\xi)=0$, $\xi\in\mathbb{R}^3\setminus O$, and any solution $V(\xi)=\rho^0\mathfrak{V}(\theta)$ is constant. Let $b>a>0$ be some numbers, and let Ξ be the annulus $\{\xi\ :\ a<\rho<b\}$. We have

$$\ln\frac{b}{a}\int_{\mathbb{S}^2}\mathfrak{Z}(\theta)ds_\theta=\int_a^b\rho^{-1}d\rho\int_{\mathbb{S}^2}\mathfrak{Z}(\theta)ds_\theta=\int_\Xi\rho^{-3}\mathfrak{Z}(\theta)d\xi=$$

$$=\int_\Xi\mathcal{D}(\nabla_\xi)^\top Y(\xi)d\xi=\int_{\mathbb{S}^2_b}\mathcal{D}(\rho^{-1}\xi)^\top Y(\xi)ds_\xi-\int_{\mathbb{S}^2_a}\mathcal{D}(\rho^{-1}\xi)^\top Y(\xi)ds_\xi=0\ .$$

We have used here the Green formula and the fact that the integrands on the spheres of radii a and b are equal to $b^{-2}\mathcal{D}(\theta)^\top\mathfrak{Y}$ and $a^{-2}\mathcal{D}(\theta)^\top\mathfrak{Y}$, respectively, i.e., the integrals cancel one another.

Therefore, the compability condition (41) is verified and the system (40) has a solution $\mathfrak{X} \in C^\infty(\mathbb{S}^2)^3$. The solution is determined up to a linear combination of traces on $\mathbb{S}^2$ of columns of the fundamental matrix $\Phi(\xi)$; recall that the columns of matrix $\Phi(\xi)$ are the only homogenuous solutions of degree -1 of the homogenuous model problem (38).

According to the definition and utility the columns Φ^q verify the relations

$$-\int\limits_{\mathbb{S}^2} \mathcal{D}(\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)\Phi^q(\xi)ds_\xi = \int\limits_{\mathbb{B}_1} L^{0j}(\nabla_\xi)\Phi^q(\xi)d\xi = \int\limits_{\mathbb{B}_1} \delta(\xi)e_q d\xi = e_q \tag{43}$$

where ξ is the unit outer normal to the sphere $\mathbb{S}^2 = \partial\mathbb{B}_1$, $\mathbb{B}_1 = \{\xi : \rho < 1\}$, δ is the Dirac mass, $e_q = (\delta_{1,q}, \delta_{2,q}, \delta_{3,q})^\top$ is the basis vector of the axis x_q, and the last integral over $\mathbb{B}_1$ is understood in the sense of the theory of distributions. Thus, owing to (43), the orthogonality condition (39) can be satisfied that implies the uniqueness of the solution $\mathfrak{X}$ to the problem (38), (39). ■

In view of (32) and (27), (28), the right-hand side of (38) takes the form

$$Z(\xi) = \mathcal{D}(\nabla_\xi)^\top(\xi^\top\nabla_\xi\mathcal{A}(P^j))\mathcal{D}(\nabla_\xi)(M^j\mathcal{D}(\nabla_\xi)\Phi^j(\xi)^\top)^\top\varepsilon^j. \tag{44}$$

General results of [6] (see also[[27]; §3.5, §6.1, §6.4]) show that there exists a unique decaying solution of problem (29), (30), which admits the expansion

$$w^{2j}(\xi) = X^j(\xi) + \Phi^j(\xi)C^j + O(\rho^{-2}(1 + |\ln|\rho||)), \quad \xi \in \mathbb{R}^3\setminus\mathbb{B}_R. \tag{45}$$

In the same way as in relation (27), the relation (45) can be differentiated term by term under the rule $\nabla_\xi O(|\rho|^{-p}(1 + |\ln\rho|)) = O(|\rho|^{-p-1}(1 + |\ln\rho|))$.

The method [14] is applied in order to evaluate the column C^j.

Lemma 3.2. *The equality is valid*

$$C^j = -\lambda(\overline{\gamma_j} - \gamma(P^j))|\omega_j|u(P^j) - I^j, \tag{46}$$

where $|\omega_j|$ is the volume, and $\overline{\gamma_j} = |\omega_j|^{-1}\int_{\omega_j}\gamma_j(\xi)d\xi$ the mean scaled density of the inclusion ω_j, i.e., its mass is $\overline{\gamma_j}\,|\omega_j|$, and

$$I^j = \int_{\mathbb{S}^2} \mathcal{D}(\xi)^\top(\xi^\top\nabla_\xi\mathcal{A}(P^j))\mathcal{D}(\nabla_\xi)(M^j\mathcal{D}(\nabla_\xi)\Phi^j(\xi)^\top)^\top ds_\xi\varepsilon^j\,. \tag{47}$$

Proof In the ball $\mathbb{B}_R$ we apply the Gauss formula and obtain, that for $R \to \infty$,

$$\begin{aligned}
\int\limits_{\mathbb{B}_R\setminus\omega_j} F^{0j}d\xi + \int\limits_{\omega_j} F^j d\xi + \int\limits_{\partial\omega_j} G^j ds_\xi &= \int\limits_{\mathbb{B}_R\setminus\omega_j} L^{j0}w^{2j}d\xi \int\limits_{\omega_j} L^j w^{2j}d\xi \\
&+ \int\limits_{\partial\omega_j} \mathcal{D}(\nu)^\top(\mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)w^{2j}_- - \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)w^{2j}_+)ds_\xi \\
&= -\int\limits_{\partial\mathbb{B}_R} \mathcal{D}(\rho^{-1}\xi)^\top\mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)w^{2j}(\xi)ds_\xi \\
= -\int\limits_{\partial\mathbb{B}_R} \mathcal{D}(R^{-1}\xi)^\top\mathcal{A}(P^j)&\mathcal{D}(\nabla_\xi)(X^j(\xi) + \Phi^j(\xi)C^j)d\xi + o(1) = C^j + o(1).
\end{aligned} \tag{48}$$

We have also taken into accout equalities (39) and (43). On the other hand, in view of formulae (36) and (32) it follows that

$$
\begin{aligned}
&\int\limits_{\omega_j} F^j(\xi)d\xi = -\lambda \int\limits_{\omega_j} \gamma_j(\xi)d\xi u(P^j) + \int\limits_{\omega_j} \mathcal{D}(\nabla_\xi)^\top \mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)U^j(\xi)d\xi \\
&\qquad = -\lambda\overline{\gamma_j}|\omega_j|u(P^j) + \int\limits_{\partial\omega_j} \mathcal{D}(\nu(\xi))^\top \mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)U^j(\xi)d\xi, \\
&\int\limits_{\mathbb{B}_R\setminus\omega_j} F^{0j}(\xi)d\xi = -\int\limits_{\partial\omega_j} \mathcal{D}(\nu(\xi))^\top(\xi^\top\nabla_\xi\mathcal{A}(P^j))\mathcal{D}(\nabla_\xi)w^{1j}(\xi)ds_\xi \\
&\qquad + \int\limits_{\partial\mathbb{B}_R} \mathcal{D}(R^{-1}\xi)^\top(\xi^\top\nabla_x\mathcal{A}(P^j))\mathcal{D}(\nabla_\xi)w^{1j}(\xi)ds_\xi.
\end{aligned}
\tag{49}
$$

We turn back to the decomposition (27), and taking into account the homogeneity degree of the integrand, we see that the integral over the sphere $\mathbb{S}^2_R = \partial\mathbb{B}_R$ equals

$$
\int\limits_{\mathbb{S}^2} \mathcal{D}(\xi)^\top(\xi^\top\nabla_x\mathcal{A}(P^j))\mathcal{D}(\nabla_\xi)(M^j\mathcal{D}(\nabla_\xi)\Phi^j(\xi)^\top)^\top ds_\xi\varepsilon^j + O(R^{-1}). \tag{50}
$$

The integrals over the surfaces $\partial\omega_j$ in the right-hand sides of (49) cancel with two integrals, which according to (33) appear in the formula

$$
\begin{aligned}
&\int\limits_{\partial\omega_j} G^j(\xi)ds_\xi = \int\limits_{\partial\omega_j} \mathcal{D}(\nu(\xi))^\top(\xi^\top\nabla_x\mathcal{A}(P^j))\mathcal{D}(\nabla_\xi)w^{1j}(\xi)ds_\xi \\
&-\int\limits_{\partial\omega_j} \mathcal{D}(\nu(\xi))^\top\mathcal{A}_{(j)}(\xi)\mathcal{D}(\nabla_\xi)U^j(\xi)d\xi + \int\limits_{\partial\omega_j} \mathcal{D}(\nu(\xi))^\top(\xi^\top\nabla_x\mathcal{A}(P^j))ds_\xi\varepsilon^j \\
&+\int\limits_{\partial\omega_j} \mathcal{D}(\nu(\xi))^\top\mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)U^j(\xi)ds_\xi.
\end{aligned}
\tag{51}
$$

Finally, by the equality

$$
\mathcal{D}(-\nabla_x)^\top\mathcal{A}^0(P^j)\mathcal{D}(-\nabla_x)U^j(\xi) + \mathcal{D}(-\nabla_x)^\top(x^\top\nabla_x\mathcal{A}^0(P^j))\varepsilon^j = \lambda\gamma^0(P^j)u(P^j)\ ,
$$

resulting from equation (33) at the point $x = P^j$, the sum of the pair of two last integrals in (51) takes the form

$$
\int_{\omega_j}(\mathcal{D}(-\nabla_\xi)^\top\mathcal{A}^0(P^j)\mathcal{D}(\nabla_\xi)U^j(\xi) + \mathcal{D}(-\nabla_\xi)^\top(\xi^\top\nabla_x\mathcal{A}^0(P^j))\varepsilon^j)d\xi = \lambda\gamma^0(P^j)|\omega_j|u(P^j)\ .
$$

It remains to pass to the limit $R \to +\infty$. ■

Now, we are in position to determine the terms v and μ in the ansätze (21) and (20), which are given by solutions of the problem

$$
\mathcal{L}(x,\nabla_x)v(x) = \lambda\gamma(x)v(x) + \mu\gamma(x)u(x) + f(x), \quad x \in \Omega\setminus\{P^1,\dots,P^J\}, \tag{52}
$$

$$
\mathcal{D}(\nu(x))^\top\mathcal{A}(x)\mathcal{D}(\nabla_x)v(x) = 0, \quad x \in \Sigma, \quad v(x) = 0, \quad x \in \Gamma. \tag{53}
$$

The weak formulation of (52)-(53) is given below by (59) in the subspace $\overset{\circ}{H}{}^1(\Omega;\Gamma)^3$ of the Sobolev space $H^1(\Omega)$. The right-hand side f includes the discrepancies, which results

from the terms of boundary layer type and of the order h^3. By decompositions (27) and (45) we obtain
(54)

$$f(x) = \sum_{j=1}^{J} (\mathcal{L}(x,\nabla_x) - \lambda\gamma(x)\mathbb{I}_3)\chi_j(x)\{(M^j\mathcal{D}(\nabla_x)\Phi^j(x-P^j)^\top)^\top\varepsilon^j + X^j(x) + \Phi^j(x-P^j)C^j\}.$$

The terms in the curly braces enjoy the singularities $O(|x-P^j|^{-2})$ and $O(|x-P^j|^{-1})$, respectively, therefore, it should be clarified in what sense the differential problem (52), (53) is considered. Equation (52) is posed in the punctured domain Ω, thus the Dirac mass and its derivatives, which are obtained by the action of the operator $\mathcal{L}$ on the fundamental matrix, are not taken into account. Beside that, by virtue of the definition of the term X^j implying a solution to the model problem (38) with the right-hand side (44), and according to the estimates of remainders in the expansions (27), (45), the following relations are valid

$$f(x) = O(r_j^{-2}(1+\ln r_j)), \quad r_j := |x-P^j| \to 0, \quad j=1,\dots,J, \tag{55}$$

which accept the differentation according to the standard rule

$$\nabla_x O(r_j^{-p}(1+|\ln r_j|))) = O(r_j^{-p-1}(1+|\ln r_j|))) \,.$$

In other words, expression (54) should be written in the combersome way

$$f(x) = \sum_{j=1}^{J} \Big\{([\mathcal{L},\chi_j] - \lambda\gamma\chi_j\mathbb{I}_3)(S^{j1}+S^{j2}) + \tag{56}$$

$$+\chi_j\mathcal{D}(\nabla_x)^\top((\mathcal{A}-\mathcal{A}(P^j)-(x-P^j)^\top\nabla_x\mathcal{A}(P^j))\mathcal{D}(\nabla_x)S^{j1} + (\mathcal{A}-\mathcal{A}(P^j))\mathcal{D}(\nabla_x)S^{j2}\Big\} \,.$$

Here, $[\mathbf{A},\mathbf{B}] = \mathbf{AB}-\mathbf{BA}$ is the commutator of operators $\mathbf{A}$ and $\mathbf{B}$, and S^{j1}, $S^{j2} = S^{j1} + X^j + \Phi^j C^j$ are expressions in curly braces in (54).

Lemma 3.3. *Let λ be a simple eigenvalue in the problem* (10), (11), *and u the corresponding vector eigenfunction normalized by the condition* (14). *Problem* (52), (53) *admits a solution $v \in H^1(\Omega)^3$ if and only if*

$$\mu = -\lim_{\delta\to 0}\int_{\Omega^\delta} u(x)^\top f(x)dx \,, \tag{57}$$

where $\Omega^\delta = \Omega\setminus(\mathbb{B}^1_\delta \cup\cdots\cup \mathbb{B}^J_\delta)$ and $\mathbb{B}^j_\delta = \{x \;:\; r_j < \delta\}$.

Proof The variant of the one dimensional Hardy's inequality

$$\int_0^1 |U(r)|^2 dr \le c\left(\int_0^1 r^2\left|\frac{dU}{dr}(r)\right|^2 dr + \int_{1/2}^1 |U(r)|^2 dr\right)$$

provides the estimate

$$\|r_j^{-1}V; L^2(\Omega)\| \le c\|V; H^1(\Omega)\|. \tag{58}$$

In this way, the last term in the integral identity serving for problem (52), (53)

$$(\mathcal{A}\nabla_x v, \nabla_x V)_\Omega - \lambda(\gamma v, V)_\Omega = \mu(\rho u, V)_\Omega + (f, V)_\Omega, \quad V \in \overset{\circ}{H}{}^1(\Omega;\Gamma)^3, \tag{59}$$

is a continuous functional over the Sobolev space $H^1(\Omega)^3$, owing to the inequalities

$$|(f,V)_\Omega| \le c\left(\|V; L_2(\Omega)\| + \sum_{j=1}^{J}\left(\int_{\mathbb{B}^j_\delta} r_j^2|f(x)|^2 dx\right)^{1/2}\left(\int_{\mathbb{B}^j_\delta} r_j^{-2}|V(x)|^2 dx\right)^{1/2}\right) \le c\|V; H^1(\Omega)\|,$$

$$\int_{\mathbb{B}^j_\delta} r_j^2 |f(x)|^2 dx \leq c \int_0^\delta r_j^2 r_j^{-2} (1 + |\ln r_j|)^2 dr_j < +\infty.$$

Thus, Lemma follows from the Riesz representation theorem and Fredholm alternative, in addition, formula (57) is valid because the integrand is a smooth function in $\Omega\setminus\{P^1, \dots, P^J\}$, with absolutely integrable singularities at the points $P^1, \dots, P^J$. ■

Remark 3.1. *If the points P^j are considered as tips of the complete cones $\mathbb{R}^3\setminus P^j$, the elliptic theory in domains with conical points (see the fundamental contributions* [6, 14, 15] *and also e.g., monograph* [27]*) provides estimates in weighted norms of the solution v to problem* (52)*,* (53)*. Indeed, owing to relation* (55) *for any $\tau > 1/2$ the inclusions $r_j^\tau f \in L^2(\mathcal{U}^j)^3$ are valid, where $\mathcal{U}^j$ stands for a neighbourhood of the point P^j, in addition $\mathcal{U}^j \cap \mathcal{U}^k = \emptyset$ for $j \neq k$, therefore, the terms $r_j^{\tau-2} v$, $r_j^{\tau-1}\nabla_x v$ and $r_j^\tau \nabla_x^2 v$ are square integrable in $\mathcal{U}^j$.* ■

We evaluate the limit in the right-hand side of (57) for $\delta \to +0$. By the Green formula and representation (54), the limit is equal to the sum of the surface integrals

(60)
$$\int_{\partial\mathbb{B}^j_\delta} \Big(S^j(x)^\top \mathcal{D}(\delta^{-1}(x-P^J))^\top \mathcal{A}(x)\mathcal{D}(\nabla_x)u(x) - u(x)^\top \mathcal{D}(\delta^{-1}(x-P^j))^\top \mathcal{A}(x)\mathcal{D}(\nabla_x) S^{j1}(x) + S^{j2}\Big) ds_x\,.$$

We apply the Taylor formulae (31) and (22) to the matrix $\mathcal{A}$ and the vector u, and take into account relations (8) for the matrices d and $\mathcal{D}$. We also introduce the stretched coordinates $\xi = \delta^{-1}(x - P^j)$. As a result, up to an infinitesimal term as $\delta \to +0$, integral (60) equals to

(61)
$$\begin{aligned}
&-\delta^{-1} I_0 + I_1 + I_2 + I_3 + I_4 + o(1)\\
&= -\delta^{-1}\int_{\mathbb{S}^2} u(P^j)^\top \mathcal{D}(\xi)^\top \mathcal{A}(P^i)\mathcal{D}(\nabla_\xi) S^{j1}(\xi) ds_\xi\\
&- \int_{\mathbb{S}^2} (d(\xi)a^j - u(P^j))^\top \mathcal{D}(\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi) S^{j1}(\xi) ds_\xi\\
&- \int_{\mathbb{S}^2} u(P^j)^\top \mathcal{D}(\xi)^\top (\xi^\top \nabla_x \mathcal{A}(P^j))\mathcal{D}(\nabla_\xi) S^{j1}(\xi) ds_\xi\\
&- \int_{\mathbb{S}^2} u(P^j)^\top \mathcal{D}(\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)(X^j(\xi) + \Phi^j(\xi)C^j) ds_\xi\\
&+ \int_{\mathbb{S}^2} (S^{j0}(\xi)^\top \mathcal{D}(\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)\mathcal{D}(\xi)^\top \varepsilon^j\\
&-(\mathcal{D}(\xi)^\top \varepsilon^j)^\top \mathcal{D}(\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi) S^{j1}(\xi)) ds_\xi + o(1).
\end{aligned}$$

The integrals I_0 and I_1 vanish. Indeed, due to the second equality in (8) we have

(62)
$$\begin{aligned}
\mathbb{R}^6 \ni &\int_{\mathbb{S}^2} d(\xi)^\top \mathcal{D}(\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi) S^{j1}(\xi) ds_\xi\\
= &-\int_{\mathbb{B}_1} d(\xi)^\top \mathcal{D}(\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)(M^j \mathcal{D}(\nabla_\xi)\Phi^j(\xi)^\top)^\top \varepsilon^j d\xi\\
= &-\int_{\mathbb{B}_1} d(\xi)^\top \mathcal{D}(\xi)^\top \delta(\xi) d\xi M^j \varepsilon^j = -(\mathcal{D}(\nabla_\xi) d(\xi))^\top|_{\xi=0} M^j \varepsilon^j = 0.
\end{aligned}$$

These equalities are understood in the sense of distributions. By formula (47), we obtain

$$I_2 = -u(P^j)^\top I^j \ .$$

Relations (39) and (43) yield

$$I_3 = u(P^j)^\top C^j \ .$$

Finally, in the same way as in (62), we obtain

$$\begin{aligned} I_4 = & \int_{\mathbb{B}_1} (\mathcal{D}(\xi)^\top \varepsilon^j)^\top \mathcal{D}(\nabla_\xi)^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi) S^{j1}(\xi) d\xi \\ = & -(\varepsilon^j)^\top \int_{\mathbb{B}_1} \mathcal{D}(\xi)\mathcal{D}(\nabla_{xi})^\top M^j \varepsilon^j \delta(\xi) d\xi = (\varepsilon^j)^\top M^j \varepsilon^j. \end{aligned} \tag{63}$$

Now, we could apply the derived formulae. We insert the obtained expressions for I_q into (61) → (60) → (57) and in view of equation (46) for the column C^j, we conclude that

$$\mu = \sum_{j=1}^{J} \Big((\varepsilon^j)^\top M^j \varepsilon^j + \lambda(\gamma(P^j) - \overline{\gamma_j})|\omega_j||u(P^j)|^2 \Big). \tag{64}$$

If equality (64) holds, then problem (52), (53) admits a solution $v \in H^1(\Omega)^3$. The construction of the detached terms in the asymptotic ansätze (20) and (21) is completed.

In the forthcoming sections the formal asymptotic analysis is confirmed and generalized into the following result.

Theorem 3.2. *Let λ_p be an eigenvalue in problem* (12) *with multiplicity $\varkappa_p$, i.e., in the sequence* (13)

$$\lambda_{p-1} < \lambda_p = \cdots = \lambda_{p+\varkappa_p-1} < \lambda_{p+\varkappa_p} \ . \tag{65}$$

There exist $h_p > 0$ and $c_p > 0$ such that for $h \in (0, h_p]$ the eigenvalues $\lambda^h_p, \cdots, \lambda^h_{p+\varkappa_p-1}$ of the singularly perturbed problem $(12)^h$*, and only the listed eigenvalues, verify the estimates*

$$|\lambda_{p+q-1} - \lambda_p - h^3 \mu^{(p)}_q| \le c_p(\alpha) h^{3+\alpha} \ , \quad q = 1, \ldots, \varkappa_p \ , \tag{66}$$

where $c_p(\alpha)$ is a multiplier depending on the number p and the exponent $\alpha \in (0, 1/2)$ but independent of $h \in (0, h_p]$, while $\mu^{(p)}_1, \cdots, \mu^{(p)}_{\varkappa_p}$ imply eigenvalues of symmetric $(\varkappa_p \times \varkappa_p)$-matrix $\mathcal{M}^p$ with the entries

(67)

$$\mathcal{M}^p_{km} = \sum_{j=1}^{J} \Big(\varepsilon(u_{p+k-1}; P^j)^\top M^j \varepsilon(u_{p+k-1}; P^j) - \lambda_p(\overline{\gamma_j} - \gamma(P^j))|\omega_j| u_{p+k-1}(P^j)^\top u_{p+k-1}(P^j) \Big) \ ,$$

M^j is the polarization matrix of the scaled inclusion (see (25) *and* (27)*), $u_{(p)}, \cdots, u_{(p+\varkappa_p-1)}$ are vector eigenfunctions in the problem* (12) *corresponding to the eigenvalue λ_p and orthonormalized by condition* (14)*, finally the quantities $\overline{\gamma_j}$ and $|\omega_j|$ are defined in Lemma 3.2.*

We explain which changes are necessary in the asymptotic ansätze (20), (21) and in the asymptotic procedure in order to construct asymptotics in the case of a multiple eigenvalue λ_p. First, for μ_p and $u_{(p)}$ in (20) and (21) should be selected unknown number $\mu^{(p)}_q$ and the linear combination

$$u^{(q)}_{(p)} = b^{(q)}_1 u_{(p)} + \cdots + b^{(q)}_{\varkappa_p} u_{(p+\varkappa_p-1)} \tag{68}$$

of vector eigenfunctions; the column $b^{(q)} = (b_1^{(q)}, \cdots, b_{\varkappa_p}^{(q)})^\top \in \mathbb{R}^{\varkappa_p}$ is of the unit norm. After the indicated changes the formulae for the boundary layers w^{1jq} and w^{2jq} remain unchanged. The same applies to problem (52), (53) for the correction term $v_{(p)}^{(q)}$ of regular type. However, the compability conditions are modified, and turn into the $\varkappa_p$ relations

$$\mu_q^{(p)}(\gamma u_{(p)}^{(q)}, u_{p+m-1})_\Omega = \lim_{\delta\to+0}\int_{\Omega^\delta} u_{p+m-1}(x)^\top f(x)dx\,, \quad m = 1,\ldots,\varkappa_p\,. \tag{69}$$

The left-hand side of (69) equals to $\mu_q^{(p)} b_m^{(q)}$ by (14) and (68). It can be evaluated by the same method as for formula (57), that (69) becomes the system of algebraic equations

$$\mu_q^{(p)} b_m^{(q)} = \sum_{k=1}^{\varkappa_p} \mathcal{M}_{mk}^{(p)} b_k^{(q)}\,, \quad m = 1,\ldots,\varkappa_p\,. \tag{70}$$

with coefficients from (67). In this way, the eigenvalues of the matrix $\mathcal{M}^{(p)}$ and its eigenvectors $b^{(q)} \in \mathbb{R}^{\varkappa_p}$ furnish the explicit values for the terms of the asymptotic ansätze (20) and (21). We emphasise that by the orthogonality and normalization conditions $(b^{(q)})^\top b^{(k)} = \delta_{q,k}$ for the eigenvectors of the symmetric matrix $\mathcal{M}^{(p)}$, it follows that the vector eigenfunctions $u_{(p)} = \left(u_{(p)}^{(1)}, \ldots, u_{(p)}^{(\varkappa_p)}\right), p = 1, \cdots, \varkappa_p$, in problem (12), which are given by formulae (68), are as well orthonormalized by the conditions (14).

If we have good luck, and from the beginning the eigenvectors $u_{(p)}, \cdots, u_{(p+\varkappa_p-1)}$ have the required form (68), then the matrix $\mathcal{M}^{(p)}$ is diagonal and the system of equations (70) is decomposed into a collection of $\varkappa_p$ independent relations, fully analoguous to relations (64) in the case of a simple eigenvalue. Such an observation is the key ingredient of the algorithm of defects identification which will be described in a forthcoming paper, and it makes the identification method insensitive to the multiplicity of eigenvalues in the limit problem.

4. Remarks on polarization matrices

The results presented in this section are borrowed from [23], and the forthcoming paper [33].

Variational formulation of problem (23) for the special fields W^{jk}, which define the elements of the polarization matrix M^j in decomposition (25), are of the form

$$\begin{aligned} 2E^j(W^{jk}, \mathcal{W}) &:= (\mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)W^{jk}, \mathcal{D}(\nabla_\xi)\mathcal{W})_{\Theta_j} + (\mathcal{A}_{(j)}\mathcal{D}(\nabla_\xi)W^{jk}, \mathcal{D}(\nabla_\xi)\mathcal{W})_{\omega_j} \\ &= (\mathcal{R}_{(j)}\mathbf{e}_k, \mathcal{D}(\nabla_\xi)\mathcal{W})_{\omega_j}, \mathcal{W} \in V_0^1(\mathbb{R}^3)^3, \end{aligned} \tag{71}$$

where $V_0^1(\mathbb{R}^3)$ is the Kondratiev space [6], which is the completion of the linear space $C_c^\infty(\mathbb{R}^3)$ (infinitely differentiable functions with compact supports) in the weighted norm

$$||W; V_0^1(\mathbb{R}^3)|| = (||\nabla_\xi W; L^2(\mathbb{R}^3)||^2 + ||(1+\rho)^{-1}W; L^2(\mathbb{R}^3)||^2)^{1/2} \tag{72}$$

The following result, established in [23, 33] can be shown by using transformations analoguous to (62) and (63) operating with the fields W^{jk} and $\mathfrak{W}^{jm} = \mathcal{D}(\xi)^\top \mathbf{e}_k + W^{jm}$.

Proposition 4.1. *The equalities hold true*

$$M_{km}^j = -2E^j(W^{jk}, W^{jm}) - \int_{\omega_j} (\mathcal{A}_{km}(P^j) - (\mathcal{A}_{(j)})_{km}(\xi))d\xi\,. \tag{73}$$

From the above representation it is clear that the matrix M^j is symmetric, the property follows by the symmetry of the stiffness matrices A^0, A^j and of the energy quadratic form E^j. In addition, the representation allows us to deduce if the matrix M^j is negative or positive definite. We write $M^1 < M^2$ for the symmetric matrices M^1 and M^2 provided all eigenvalues of $M^2 - M^1$ are positive.

Proposition 4.2. (see [33]) **1°** *If $\mathcal{A}_{(j)}(\xi) < \mathcal{A}(P^j)$ for $\xi \in \omega_j$ (the inclusion is softer compared to the matrix material), then M^j is a negative definite matrix.*
2° *If the matrix $\mathcal{A}_{(j)}$ is constant and $\mathcal{A}_{(j)}^{-1} < \mathcal{A}(P^j)^{-1}$ (the homogenuous inclusion is rigid compared to the matrix), then M^j is a positive definite matrix.*

It is also possible to consider the limit cases, either of a cavity with $\mathcal{A}^j = 0$, or of an absolutely stiff inclusion with $\mathcal{A}_{(j)} = \infty$. For the case of a cavity the diifferential problem takes the form

$$
\begin{aligned}
&L^{0j}(\nabla_\xi)W^{jk}(\xi) = 0, \quad \xi \in \Theta_j = \mathbb{R}^3 \setminus \overline{\omega_j}, \\
&\mathcal{D}(\nu(\xi))^\top \mathcal{A}(P^j)\mathcal{D}(\nabla_\xi)W^{jk}(\xi) = -\mathcal{D}(\nu(\xi))^\top \mathcal{A}(P^1)\mathbf{e}_k, \quad \xi \in \partial\omega_j\,.
\end{aligned}
\tag{74}
$$

For an absolutely rigid inclusion the integral-differential equations occur as follows

$$
\begin{aligned}
&L^{0j}(\nabla_\xi)W^{jk}(\xi) = 0, \quad \xi \in \Theta_j, \quad W^{jk}(\xi) = d(\xi)c^{jk} - \mathcal{D}(\xi)^\top \mathbf{e}_k, \quad \xi \in \partial\omega_j, \\
&\int_{\partial\omega_j} d(\xi)^\top \mathcal{D}(\nu(\xi))^\top \mathcal{A}(P^j)(\mathcal{D}(\nabla_\xi)W^{jk}(\xi) - \mathbf{e}_k)ds_\xi = 0 \in \mathbb{R}^6\,,
\end{aligned}
\tag{75}
$$

where the matrices $\mathcal{D}$ and d are introduced in (3) and (7), respectively.

The Dirichlet conditions in (75) contains an arbitrary column $c^{jk} \in \mathbb{R}^6$, which permits for rigid motion of ω_j and can be determined by the integral conditions which annulate the principal vector and moment of forces applied to the body ω_j. The variational formulation of problems (74) and (75) can be established in the Kondratiev space $V_0^1(\Theta_j)^3$ (see [6], and e.g., [27]) normed by the weighted norm (72) (cf. the right-hand side of (76)) and in its linear subspace $\{W \in V_0^1(\Theta_j)^3 : W\big|_{\partial\omega_j} \in \mathcal{R}\}$, respectively, where $\mathcal{R}$ is the linear space of rigid motions (6). The asymptotic procedures of derivation of problems (74) and (75) from problems (23) and (71) can be found in [38, 13]. The required estimates can be extracted from these references as well.

In accordance with Proposition 4.2 the polarization matrix for a cavity is always negative definite, and that for an absolutely rigid inclusion, it is always positive definite. Theorem 3.2 gives an asymptotic formula, which can be combined with the indicated facts and the information from Proposition 4.2, and it makes possible to deduce the sign of the variation of a given eigenvalue in terms of the defect properties. For example, in the case of a defect-crack, with the null volume and negative polarization matrix, the eigenvalues of the weakened body are smaller compared to the initial body. Such an observation is already employed in the bone China porcelane shops by the qualified personel.

5. Justification of asymptotics

We proceed with the following statements, which are fairly known for the entire body (see [34, 7]) but should be verified for a body with small cavities (see (16)). We emphasize that a body with small inclusions is to be regarded in some sense as an intermediate case. In this way, some of given below axiliary results for the intact body are fit for the body with foreign inclusions, however, in some situations it is much simpler to compare the latter with the body with small voids. On the other hand, the whole justification procedure works for any sort of defects.

Proposition 5.1. *For a vector function $u \in \overset{\circ}{H}{}^1(\Omega;\Gamma)^3$ the inequality*

$$\|r_j^{-1}u; L^2(\Omega)\| + \|\nabla_x u; L^2(\Omega)\| \leq c\|\mathcal{D}(\nabla_x)u; L^2(\Omega)\| \tag{76}$$

holds true. The above inequality remains valid with a constant independent of $h \in (0, h_0]$, if the domain Ω is replaced by the domain $\Omega(h)$ with defects.

Proof For analysis of displacement fields in the domain $\Omega(h)$ with cavities (in particular, with cracks) we apply the method described in review papier [[29]; §2.3] - in this framework the body with elastic inclusions is considered as intact or entire. Let us consider the restriction $\widehat{u}$ of u to the set $\Omega^h = \Omega \setminus \bigcup_{j=1}^J \overline{\mathbb{B}_{hR}^j}$, where $\mathbb{B}_{hR}^j = \{x : |x - P^j| < hR\}$ and radius hR of the balls is selected in such a way that $\overline{\omega_j^h} \subset \mathbb{B}_{hR}^j$. We construct an extension $\widetilde{u}$ to Ω of the field $\widehat{u}$. To this end, we introduce the annulae $\Xi_h^j = \mathbb{B}_{2hR}^j \setminus \overline{\mathbb{B}_{hR}^j}$ and perform the stretching of coordinates $x \mapsto \xi^j = h^{-1}(x - P^j)$. The vector functions $\widehat{u}$ and u written in the ξ^j-coordinates are denoted by $\widehat{u}^j$ and u^j, respectively. It is evident that

$$h\|\mathcal{D}(\nabla_\xi)\widehat{u}^j; L^2(\Xi)\|^2 = \|\mathcal{D}(\nabla_\xi)\widehat{u}; L^2(\Xi_h^j)\|^2 \leq \|\mathcal{D}(\nabla_x)u; L^2(\Omega(h))\|^2; \tag{77}$$

where $\Xi = \mathbb{B}_{2R} \setminus \overline{\mathbb{B}_R}$. Let

$$\widehat{u}^j(\xi^j) = \widehat{u}_\perp^j(\xi^j) + d(\xi^j)a^j, \tag{78}$$

where d is the matrix (7), and the column $a^j \in \mathbb{R}^6$ is selected in such a way that

$$\int_\Xi d(\xi^j)^\top \hat{u}_\perp^j(\xi^j)d\xi^j = 0 \in \mathbb{R}^6, \tag{79}$$

where the matrix d is given by (7). By the orthogonality condition (79), the Korn inequality is valid

$$\|\widehat{u}_\perp^j; H^1(\Xi)\| \leq c_R\|\mathcal{D}(\nabla_\xi)\widehat{u}_\perp^j; L^2(\Xi)\| = c_R\|\mathcal{D}(\nabla_\xi)\widehat{u}^j; L^2(\Xi)\| \tag{80}$$

(see, e.g., [7], [[29]; §2] and [[24]; Thm 2.3.3]), and the last equality follows from the second formula (8) since the rigid motion da^j generates null strains (4). Let $\widetilde{u}_\perp^j$ denote an extension in the Sobolev class H^1 of the vector function $\widehat{u}_\perp^j$ from Ξ onto $\mathbb{B}_R$, such that

$$\|\widehat{u}_\perp^j; H^1(\mathbb{B}_{2R})\| \leq c_R\|\widehat{u}_\perp^j; H^1(\Xi)\|. \tag{81}$$

Now, the required extension of the field u onto the entire domain Ω is given by the formula

$$\widetilde{u}(x) = \begin{cases} \widehat{u}(x), & x \in \Omega^h\,, \\ d(\xi^j)a^j + \widetilde{u}_\perp^j(\xi^j), & x \in \mathbb{B}_{hR}^j, \quad j = 1, \ldots, J. \end{cases} \tag{82}$$

In addition, according to (78) and (77), (80), (81) we have

$$\|\mathcal{D}(\nabla_x)\widetilde{u}; L^2(\Omega)\| \leq c\|\mathcal{D}(\nabla_x)u; L^2(\Omega(h))\|. \tag{83}$$

Applying the Korn's inequality (80) in Ω, we obtain

$$\|r_j^{-1}u; L^2(\Omega^h)\| + \|\nabla_x u; L^2(\Omega^h)\| \leq \|r_j^{-1}\widetilde{u}; L_2(\Omega)\| + \|\nabla_x\widetilde{u}; L_\Omega\| \leq c\|\mathcal{D}(\nabla_x)\widetilde{u}; L^2(\Omega)\|. \tag{84}$$

We turn back to the function $\widehat{u}^j$ and find

$$h\|\widehat{u}^j; H^1(\Xi)\|^2 \leq c(\|r_j^{-1}\widetilde{u}; L^2(\Omega)\|^2 + \|\nabla_x\widetilde{u}; L^2(\Omega)\|^2). \tag{85}$$

The other variant of the Korn's inequality

$$\|u^j; H^1(\mathbb{B}_{2R} \setminus \omega_j)\|^2 \leq c(\|\mathcal{D}(\nabla_x)u^j; L^2(\Xi \setminus \omega_j)\|^2 + \|u^j; L^2(\Xi)\|^2) \tag{86}$$

(see e.g., [7], [[29]; §2] or [[24]; §3.1]), after returning to the x-coordinates leads to the relations

$$h^{-2}\|u; L^2(\mathbb{B}_{2hR}\setminus\omega_j^h)\|^2 \leq c\|\nabla_x u; L^2(\mathbb{B}_{2hR}\setminus\omega_j^h)\|^2 \tag{87}$$
$$\leq c(\|\mathcal{D}(\nabla_x)u; L^2(\mathbb{B}_{2hR}\setminus\omega_j^h)\|^2 + h^{-2}\|u; L^2(\Xi_{hR}^j)\|^2)$$

By virtue of $Ch \geq r_j \geq ch > 0$ for $x \in \mathbb{B}_{2hR}\setminus\omega_j^h \supset \Xi_{hR}^j$, the multiplier h^{-1} can be inserted into the norm, and transformed to r_j^{-1}, but the norm $\|r_j^{-1}u; L^2(\Xi_{hR}^j)\|$ is already estimated in (84), owing to $\widetilde{u} = u$ on Ξ_{hR}^j. Estimates (87), $j = 1, \ldots, J$, modified in the indicated way along with relation (84) imply the Korn inequality in the domain $\Omega(h)$. ■

Remark 5.1. *If ω_j is a domain, then in the proof of Proposition 5.1 we do not need to restrict $\widehat{u}$ to Ω^h, but operate directly with the sets $\Omega(h)$ and $\mathbb{B}_{2R}\setminus\omega_j$ since there is a bounded extension operator in the class H^1 over the Lipschitz boundary $\partial\omega_j$ with the estimate of type* (81)*. The presence of cracks $\overline{\omega_j^h}$ makes the existence of such an extension impossible. However, the Korn's inequality* (87) *is still valid in this case, since to maintain the valididity the union of Lipschitz domains is only required (see* [7]*).* ■

The bilinear form

$$\langle u, v\rangle = (\mathcal{A}^h\mathcal{D}(\nabla_x)u, \mathcal{D}(\nabla_x)v)_\Omega \tag{88}$$

can be taken as a scalar product in the Hilbert space $\overset{\circ}{H}{}^1(\Omega;\Gamma)^3$. In this way, the integral identity $(12)^h$ can be rewritten as the abstract spectral equation

$$\mathfrak{K}^h u^h = \mathfrak{m}^h u^h\ , \tag{89}$$

where $\mathfrak{m}^h = (\lambda^h)^{-1}$ is the new spectral parameter, and $\mathfrak{K}^h$ is a compact, symmetric, and continuous operator, thus selfadjoint,

$$\langle\mathfrak{K}^h u, v\rangle = (\gamma^h u, v)_\Omega\ , \quad u, v \in \mathfrak{H}\ . \tag{90}$$

Eigenvalues of the operator $\mathfrak{K}^h$ constitute the sequence

$$\mathfrak{m}_1^h \geq \mathfrak{m}_2^h \geq \cdots \geq \mathfrak{m}_p^h \geq \cdots \to +0\ , \tag{91}$$

with the elements related to the sequence in (18) by the first formula in (90).

The following statement is known as Lemma on *almost eigenvalues and eigenvectors* (see, e.g., [42]).

Proposition 5.2. *Let $\mathfrak{m}$ and $\mathfrak{u} \in \mathfrak{H}$ be such that*

$$\|\mathfrak{u}\|_{\mathfrak{H}} = 1\ , \quad \|\mathfrak{K}^h\mathfrak{u} - \mathfrak{m}\mathfrak{u}\|_{\mathfrak{H}} = \delta\ . \tag{92}$$

Then there exists an eigenvalue $\mathfrak{m}_p^h$ of the operator $\mathfrak{K}^h$, which satisfies the inequality

$$|\mathfrak{m} - \mathfrak{m}_p^h| \leq \delta. \tag{93}$$

Moreover, for any $\delta_\bullet > \delta$ the following inequality holds

$$\|\mathfrak{u} - \mathfrak{u}_\bullet\|_{\mathfrak{H}} \leq 2\delta/\delta_\bullet \tag{94}$$

where $\mathfrak{u}_\bullet$ is a linear combination of eigenfunctions of the operator $\mathfrak{K}^h$, associated to the eigenvalues from the segment $[\mathfrak{m} - \delta_\bullet, \mathfrak{m} + \delta_\bullet]$, and $\|\mathfrak{u}_\bullet\|_{\mathfrak{H}} = 1$.

For the asymptotic approximations $\mathfrak{m}$ and $\mathfrak{u}$ of solutions to the abstract equation (89) we take

$$\mathfrak{m} = (\lambda_p + h^3\mu_p)^{-1}\ , \mathfrak{u} = \|U; \mathfrak{H}\|^{-1}U\ , \tag{95}$$

where U stands for the sum of terms separated in the asymptotic ansatz (21). Let us evaluate the quantity δ from formula (92). By virtue of $\lambda_p > 0$, for $h \in (0, h_p]$ and $h_p > 0$ small enough, we have

$$\begin{aligned}(96)\qquad \delta &= \|\mathfrak{K}^h \mathfrak{u} - \mathfrak{m}\mathfrak{u}; \mathfrak{H}\| = (\lambda_p + h^3\mu_p)^{-1}\|U; \mathfrak{H}\|^{-1} \sup_{v\in\mathfrak{S}} |(\lambda_p + h^{3\mu_p})\langle \mathfrak{K}^h U, \rangle - \langle U, V\rangle| \\ &\le c \sup_{v\in\mathfrak{S}} |(\mathcal{A}^h \mathcal{D}(\nabla_x)U; \mathcal{D}(\nabla_x)V)_\Omega - (\lambda_p + h^3\mu_p)(\rho^h U^h, V)_\Omega|;\end{aligned}$$

where $\mathfrak{S} = \{V \in \mathfrak{H} : \|V; \mathfrak{H}\| = 1\}$ is the unit sphere in the space $\mathfrak{H}$. In addition, to estimate the norm $\|U; \mathfrak{H}\|$ the following relations are used

$$\begin{aligned}(97)\qquad &\|u_{(p)}; \mathfrak{H}\|^2 = (\mathcal{A}^h \mathcal{D}(\nabla_x)u_{(p)}, \mathcal{D}(\nabla_x)u_{(p)})_\Omega \ge c > 0, \\ &\|h^i \chi_j w^{ij}_{(p)}; \mathfrak{H}\| \le ch^{i+1/2}, \quad i = 1, 2, \quad \|h^3 v_{(p)}; \mathfrak{H}\|^2 \le ch^3,\end{aligned}$$

where the first relation follows from the continuity at the points P^j of the second order derivatives of the vector function $u_{(p)}$ combined with the integral identity (12) and the normalization condition (19). We transform the expression under the sign *sup* in (96). Substituting into the expression the sum of terms in ansatz (21), we have

$$\begin{aligned}(98)\qquad \mathbf{I}_0 &= (\mathcal{A}^h \mathcal{D}(\nabla_x)u_{(p)}, \mathcal{D}(\nabla_x)V)_\Omega - (\lambda_p + h^3\mu_p)(\gamma^h u_{(p)}, V)_\Omega \\ &= \textstyle\sum_{j=1}^J \left\{((\mathcal{A}_{(j)} - \mathcal{A})\mathcal{D}(\nabla_x)V)_{\omega_j^h} - \lambda_p((\gamma^h - \gamma)u_{(p)}, V)_{\omega_j^h}\right\} \\ &\quad -h^3\mu_p(\gamma^h u_{(p)}, V)_\Omega =: \textstyle\sum_{j=1}^J \mathbf{I}_0^j - \mathbf{I}_0^0,\end{aligned}$$

$$(99)\quad \mathbf{I}_i^j = h^i(\mathcal{A}\mathcal{D}(\nabla_x)\chi_j w^{ij}_{(p)}, \mathcal{D}(\nabla_x)V)_\Omega - h^i(\lambda_p + h^3\mu_p)(\gamma^h \chi_j w^{ij}_{(p)} V)_\Omega = \mathbf{I}_i^{j0} - \mathbf{I}_i^{j0}, \quad i = 1, 2,$$

$$\begin{aligned}(100)\qquad \mathbf{I}_4 &= h^3((\mathcal{A}\mathcal{D}(\nabla_x)v_{(p)}\mathcal{D}(\nabla_x)V)_\Omega - \lambda_p(\gamma v_{(p)}, V)_\Omega) - h^6\mu_p(\gamma^h v, V)_\Omega \\ &\quad + h^3 \textstyle\sum_{j=1}^J \left\{((\mathcal{A}_{(j)} - \mathcal{A})\mathcal{D}(\nabla_x)v_{(p)}, \mathcal{D}(\nabla_x)V)_{\omega_j^h} - \lambda_p((\gamma_j - \gamma)v_{(p)}, V)_{\omega_j^h}\right\} \\ &= h^3\mathbf{I}_4^0 + h^6\mathbf{I}_4^{01} + h^3 \textstyle\sum_{j=1}^J \mathbf{I}_4^j.\end{aligned}$$

In (98) we used that $u_{(p)}$ and λ_p verify the integral identity (12). Furthermore, by the Taylor formulae (34) and (31), we obtain

(101)

$$\begin{aligned}|\mathbf{I}_0^j - \mathbf{I}_0^{j1} - \mathbf{I}_0^{j2}| &\le c(h^2\|\mathcal{D}(\nabla_x)V; L^1(\omega)_j^h\| + h\|V; L^1(\omega_j^h)\| \\ &\quad + \int_{\omega_j^h} |V - \overline{V}^j| dx) \le ch^2h^{3/2}\|\mathcal{D}(\nabla_x)V; L^2(\Omega)\| = ch^{7/2}, \\ \mathbf{I}_0^{j1} &= ((\mathcal{A}_{(j)} - \mathcal{A}(P^j))\varepsilon^j, \mathcal{D}(\nabla_x)V)_{\omega_h^j}, \\ \mathbf{I}_0^{j2} &= ((\mathcal{A}_{(j)} - \mathcal{A}(P^j))\mathcal{D}(\nabla_x)U^j_{(p)}, \mathcal{D}(\nabla_x)V)_{\omega_h^j} + ((x - P^j)^\top \nabla_x \mathcal{A}(P^j)\varepsilon^j_{(p)}\mathcal{D}(\nabla_x)V)_{\omega_h^j} \\ &\quad -\lambda_p((\gamma_j - \gamma(P^j))u_{(p)}(P^j), V)_{\omega_j^h}.\end{aligned}$$

Let explain the derivation of above formulae. The following substitutions are performed

$$\begin{gathered}\mathcal{D}(\nabla_x)u_{(p)}(x) \mapsto \varepsilon^j_{(p)} + \mathcal{D}(\nabla_x)U_{(p)}(x), \\ \mathcal{A}(x) \mapsto \mathcal{A}(P^j) + (x - P^j)^\top \nabla_x \mathcal{A}(P^j), \\ u_{(p)}(x) \mapsto u_{(p)}(P^j),\end{gathered}$$

with pointwise estimates for remainders of orders h^2, h^2, and h, respectively. These gave rise to the following multipliers in the majorants

$$\|\mathcal{D}(\nabla_x)V; L^1(\omega_h^j)\| \leq ch^{3/2}\|\mathcal{D}(\nabla_x)V; L^2(\Omega)\|,$$
$$\|V; L^1(\omega_h^j)\| \leq ch^{3/2}\|r_j^{-1}V; L^2(\Omega)\| .$$

Note that the factor $h^{3/2}$ is proportional to $(\mathrm{mes}_3\omega_j^h)^{1/2}$, and $h^{-1}r_j$ does not exceed a constant on the inclusion ω_j^h. Beside that, the Poincaré inequality

$$\int\limits_{\omega_j^h} |V(x) - \overline{V}^j|dx \leq ch^{3/2} \int\limits_{\omega_j^h} |V(x) - \overline{V}^2|^2 dx \leq ch^{3/2}h^2 \int\limits_{\omega_j^h} |\nabla_x V(x)|^2 dx, \tag{102}$$

is employed together with the relation

$$\int\limits_{\omega_j} (\gamma_j(x) - \overline{\gamma}_j)u_{(p)}(P^j)^\top V(x)dx = \int\limits_{\omega_j} (\gamma_j(x) - \overline{\gamma}_j)u_{(p)}(P^j)^\top (V(x) - \overline{V}^j)dx. \tag{103}$$

Here $\overline{V}^j$ stands for the mean value of V over ω_j^h. Finally, all the norms of the test function V are estimated by Proposition 5.1.

In similar but much simpler way, by virtue of Remark 3.1, the term $\mathbf{I}_4^j$ from (100) satisfies

$$h^3|\mathbf{I}_4^j| \leq ch^3(h^{1-\tau}\|r_j^{\tau-1}\nabla_x v_{(p)}; L^2(\omega_j^h)\| + h^{2-\tau}\|r_j^{\tau-2}v_{(p)}; L^2(\omega_j^h)\|)\|V; \mathfrak{H}\| \leq ch^{4-\tau}, \tag{104}$$

where $\tau > 1/2$ is arbitrary. It is clear that $h^6|\mathbf{I}_4^{01}| \leq Ch^6$. The integral $h^3\mathbf{I}_4^0$ cancels the integral $-h^3\mathbf{I}_0^0$ in (98) and some parts of the integrals $\mathbf{I}_i^j$ from (99), which we are going to consider.

In the notation of formula (56) we have

$$\begin{aligned}
\mathbf{I}_i^j &= h^i \Big\{(\mathcal{A}_{(j)}\mathcal{D}(\nabla_x)w_{(p)}^{ij}, \mathcal{D}(\nabla_x)V)_{\omega_j^h} + (\mathcal{A}(P^j)\mathcal{D}(\nabla_x)w_{(p)}^{ij}, \mathcal{D}(\nabla_x)\chi_j V)_{\Omega\setminus\omega_j^h} \\
&\quad + h^{-1}\delta_{i,2}((x - P^j)^\top \nabla_x \mathcal{A}(P^j)\mathcal{D}(\nabla_x)w_{(p)}^{1j}, \mathcal{D}(\nabla_x)\chi_j V)_{\Omega\setminus\omega_j^h}\Big\} \\
&\quad + \Big\{(\mathcal{A}[\mathcal{D}(\nabla_x), \chi_j]w_{(p)}^{ij}, \mathcal{D}(\nabla_x)V)_\Omega - (\mathcal{A}\mathcal{D}(\nabla_x)w_{(p)}^{ij}, [\mathcal{D}(\nabla_x), \chi_j]V)_\Omega\Big\} \\
&\quad + ((\mathcal{A} - \mathcal{A}(P^j) - \delta_{i,1}(x - P^j)^\top \nabla_x \mathcal{A}(P^j))\mathcal{D}(\nabla_x)w_{(p)}^{ij}, \mathcal{D}(\nabla_x)\chi_j V)_{\Omega\setminus\omega_j^h} \\
&=: h^i\mathbf{I}_i^{j0} + \mathbf{I}_i^{j1} + \mathbf{I}_i^{j2}.
\end{aligned} \tag{105}$$

Furthermore, the integrals $h^i\mathbf{I}_i^{j0}$ and $\mathbf{I}_i^{ji}$ cancel each other according to the integral identities

$$\begin{aligned}
2E^j(w^{1j}, \chi_j V) &= ((\mathcal{A}(P^j) - \mathcal{A}_{(j)})\varepsilon_p^j, \mathcal{D}(\nabla_\xi)\chi_j V)_{\omega_j}, \\
2E^2(w^{2j}, \chi_j V) &= (F^{0j}, \chi_j V)_{\mathbb{R}^3\setminus\omega_j} + (F^j, V)_{\omega_j} + (G^j, V)_{\partial\omega_j} .
\end{aligned} \tag{106}$$

The latter formulae are provided by (71), (26) and (29), (30), (32), (33), (36). We point out that the test function $\xi \mapsto \chi_j(h\xi + P^j)\mathcal{V}(h\xi + P^j)$ in (106) has a compact support, i.e., the function belongs to the Kondratiev space $V_0^1(\mathbb{R}^3)$, and in the analysed integrals the stretching of coordinates $x \mapsto \xi = h^{-1}(x - P^j)$ has to be performed.

The expressions including asymptotic terms $S_{(p)}^{ji}(h^{-1}(x - P^j)) = h^{3-i}S_{(p)}^{ji}(x - P^j)$ are detached from the integrals $\mathbf{I}_i^{j1}$ and $\mathbf{I}_i^{j2}$,

$$\begin{aligned}
\mathbf{I}_{i0}^{j1} &= h^3\Big\{(\mathcal{A}[\mathcal{D}(\nabla_x), \chi_j]S_{(p)}^{ji}, \mathcal{D}(\nabla_x)V)_\Omega - (\mathcal{A}\mathcal{D}(\nabla_x)S_{(p)}^{ji}[\mathcal{D}(\nabla_x), \chi_j]V)_\Omega\Big\} \\
&= h^3([\mathcal{L}, \chi_j]S^{ji}, V)_\Omega, \\
\mathbf{I}_{i0}^{j2} &= h^3((\mathcal{A} - \mathcal{A}(P^j) - \delta_{i,1}(x - P^j)^\top \nabla_x \mathcal{A}(P^j))\mathcal{D}(\nabla_x)S_{(p)}^{ji}, \mathcal{D}(\nabla_x)\chi_j V)_{\Omega\setminus\omega_j^h},
\end{aligned} \tag{107}$$

and the remainders are estimated by virtue of the decompositions (27) and (45), namely,

(108)
$$
\begin{aligned}
|\mathbf{I}_1^{j1}-\mathbf{I}_{10}^{j1}| &\leq ch\|V;\mathfrak{H}\|\left(\int\limits_{\sup|\nabla_x\chi_j|}((1+h^{-1}r_j)^{-6}+h^{-2}(1+h^{-1}r_j)^{-8})dx\right)^{1/2}\leq ch^4,\\
|\mathbf{I}_1^{j2}-\mathbf{I}_{10}^{j2}| &\leq ch^2\|V;\mathfrak{H}\|\left(\int\limits_{\sup|\nabla_x\chi_j|}((1+h^{-1}r_j)^{-4}+h^{-2}(1+h^{-1}r_j)^{-6})(1+|\ln(h^{-1}r_j)|)^2dx\right)^{1/2}\\
&\leq ch^4(1+|\ln h|),\\
|\mathbf{I}_2^{j1}-\mathbf{I}_{20}^{j1}| &\leq ch\|V;\mathfrak{H}\|\left(\int\limits_{\Omega\setminus\omega_j^h} r_j^4(1+h^{-1}r_j)^{-6}dx\right)^{1/2}\leq ch^4,\\
|\mathbf{I}_2^{j2}-\mathbf{I}_{20}^{j2}| &\leq ch^2\|V;\mathfrak{H}\|\left(\int\limits_{\Omega\setminus\omega_j^h} r_j^2(1+h^{-1}r_j)^{-4}(1+|\ln(h^{-1}r_j)|)^2dx\right)^{1/2}\leq ch^4(1+|\ln h|).
\end{aligned}
$$

Inequalities for the integrals $\mathbf{I}_i^{j0}$ from (99) are obtained in a similar way and look as follows :

(109)
$$
\begin{aligned}
&|\mathbf{I}_i^{j0}-\mathbf{I}_{i0}^{j0}|\leq c\|r_j^{-1}V;L^2(\Omega)\|h^ih^{4-i}(1+\delta_{i,2}|\ln h|)\leq ch^4(1+\delta_{i,2}|\ln h|),\\
&\mathbf{I}_{i0}^{j0}=h^3\lambda_{(p)}(\rho\chi_jS_{(p)}^{ji},V)_\Omega\ .
\end{aligned}
$$

According to formula (56) for the right-hand side f of the problem (52), (53) and the associated integral identity (59), the sum of the expressions $h^3\mathbf{I}_4^0$ from (100) and $\mathbf{I}_{i0}^{iq}$ from (107), (109) (the latter is summed over $j=1,\dots,J$ and $q=0,1,2$) turns out to vanish. As a result, collecting the obtained estimates, we conclude that the quantity δ from formula (95) (see also (92)) satisfies the estimate

$$\delta\leq c_\alpha h^{3+\alpha} \tag{110}$$

for any $\alpha\in(0,1/2)$.

Now we are in position to prove the main theorem on asymptotics of solutions of singularly perturbed problem.

Proof of Theorem 3.2 From the columns $b^{(1)},\dots,b^{(\varkappa_p)}$ of matrix $\mathcal{M}^{(p)}$ with elements (67) can be constructed linear combinations (68) of vector eigenfunctions $u_{(p)},\dots,u_{(p+\varkappa_p-1)}$ as well as the subsequent terms of asymptotic ansatz (21). As a result, for $q=p,\dots,p+\varkappa_p-1$ the approximate solutions $\left\{(\lambda_p+h^3\mu_p)^{-1}\ ,\|U_{(p)}^{(q)};\mathfrak{H}\|^{-1}U_{(p)}^{(q)}\right\}$ of the abstract equation (89) are obtained, such that the quantity δ from relations (92) verifies the inequality (110). We apply the second part of Proposition 5.2 with

$$\delta_\bullet=c_\bullet h^{3+\alpha_\bullet}\ ,\quad \alpha_\bullet\in(0,\alpha)\ . \tag{111}$$

Let the list

$$\mathfrak{m}_n^h=(\lambda_n^h)^{-1},\cdots,\mathfrak{m}_{n+N-1}^h=(\lambda_{n+N-1}^h)^{-1} \tag{112}$$

include all eigenvalues of the operator $\mathfrak{K}^h$, located in the segment

$$[(\lambda_p)^{-1}-c_\bullet h^{3+\alpha_\bullet},(\lambda_p)^{-1}+c_\bullet h^{3+\alpha_\bullet}]\ , \tag{113}$$

for sufficiently small $h_\bullet > 0$, such that $(\lambda_p + h^3\mu_p)^{-1}$ with $h \in (0, h_\bullet]$ belongs to segment (113). Our immediate objective becomes to show that

$$(114) \qquad n = p, \quad N = \varkappa_p \, .$$

The quantities $\mathfrak{m}_n^h$ for $m \geq n+N-1$ are uniformly bounded in $h \in (0, h_\bullet]$. By Proposition 5.1, the same assumptions provide the uniform boundedness of the norm $\|\widetilde{u}_{(m)}^h; \overset{\circ}{H}{}^1(\Omega;\Gamma)^3\|$ of the vector functions $\widetilde{u}_{(m)}^h \in \mathfrak{H}^h$ constructed for the vector eigenfunctions $u_{(m)}^h$ in $(12)^h$ according to (86). Hence, there exists an infinitesimal sequence $\{h_i\}$, such that the limit passage $h_i \to +0$ leads to the convergences

$$(115) \quad \mathfrak{m}_m^h \to \mathfrak{m}_m^0 = (\lambda_m^0)^{-1} , \quad \widetilde{u_{(m)}^h} \to \widetilde{u_{(m)}^0} \quad \text{weakly in } H^1(\Omega)^3 \quad \text{and strongly in } L^2(\Omega)^3 \, .$$

We substitute into the integral identity $(12)^h$ the test function $v \in C_c^\infty(\overline{\Omega}\diagdown(\Gamma\cup\{P^1, \cdots, P^J\}))^3$. According to definition (17) and for sufficiently small $h > 0$, the stiffness matrix $\mathcal{A}^h$ and the density γ^h coincide on the support of v with $\mathcal{A}$ and γ, respectively. Therefore, the limit passage $h_i \to +0$ in the integral identity $(12)^h$ leads to the equality

$$(116) \qquad (\mathcal{A}\mathcal{D}\, \widetilde{u_{(m)}^0}, \mathcal{D}v)_\Omega = \lambda_m^0 (\gamma\, \widetilde{u_{(m)}^0}, v)_\Omega \, .$$

Since $C_c^\infty(\overline{\Omega}\diagdown(\Gamma \cup \{P^1, \cdots, P^J\}))^3$ is dense in $\overset{\circ}{H}{}^1(\Omega;\Gamma)^3$, the integral identity (116) holds true for all test functions $v \in \overset{\circ}{H}{}^1(\Omega;\Gamma)^3$. We observe that the weighted norms $\|r_j^{-1}\widetilde{u_{(m)}^h}; L^2(\Omega)\|$ are uniformly bounded by virtue of inequality (76), thus

$$(\gamma^h\, \widetilde{u_{(m)}^h}, \widetilde{u_{(l)}^h})_\Omega - (\gamma\, \widetilde{u_{(m)}^h}, \widetilde{u_{(l)}^h})_\Omega = o(1) \quad \text{for } h \to +0 \, .$$

In this way, taking into account formulae (19) and (115), we find out that

$$(117) \qquad (\gamma\, \widetilde{u_{(m)}^0}, \widetilde{u_{(l)}^0})_\Omega = \delta_{m,l} \, .$$

Hence, λ_m^0 is an eigenvalue, and $\widetilde{u_{(m)}^0}$ is a normalized vector eigenfunction of the limit problem (12). This implies that $p+\varkappa_p \geq n+N$. Considering consenquently the eigenvalues $\lambda_p, \ldots, \lambda_1$, we conclude that

$$(118) \qquad p \geq n \, , \quad \varkappa_p \geq N \, .$$

In order to establish the inequalities $p \leq n$ and $\varkappa_p \leq N$ we select the factor $c_\bullet$ in (111) such that for $\mu_p^{(k)} \neq \mu_p^{(q)}$ the number $(\lambda_p + h^3\mu_p^{(k)})^{-1}$ is excluded from the segment

$$(119) \qquad [(\lambda_p + h^3\mu_p^{(q)})^{-1} - c_\bullet h^{3+\alpha_\bullet}, (\lambda_p + h^3\mu_p^{(q)})^{-1} + c_\bullet h^{3+\alpha_\bullet}] \, .$$

Let $\kappa_p^{(q)}$ be the multiplicity of the eigenvalue $\mu_p^{(q)}$ of matrix $\mathcal{M}^{(p)}$. By Proposition 5.1 and estimate (120) there are, not necessarily distinct, eigenvalues $\mathfrak{m}_{l(q)}^h, \ldots, \mathfrak{m}_{l(q+\kappa_q-1)}^h$ of the operator $\mathfrak{K}^h$ such that

$$(120) \qquad |\mathfrak{m}_{l(k)}^h - (\lambda_p + h^3\mu_p^{(q)})^{-1}| \leq c_{pq}^\alpha h^{3+\alpha} \, .$$

In addition, Proposition 5.1 furnishes the normalized columns $\mathfrak{a}^{(k)} = (\mathfrak{a}_{n_\bullet}^{(k)}, \cdots, \mathfrak{a}_{n_\bullet+N_\bullet-1}^{(k)})^\top$, such that

$$(121) \qquad \left\| U_{(p)}^{(k)} - \|U_{(p)}^{(k)}; \mathfrak{H}\| \sum_{i=1}^{n_\bullet+N_\bullet-1} \mathfrak{a}_i^{(k)} \mathfrak{u}_i^{(h)}; \mathfrak{H} \right\| \leq \frac{\delta}{\delta_\bullet} \leq \frac{c}{c_\bullet} h^{\alpha-\alpha_\bullet} \, ,$$

where $\mathfrak{u}^h_{n_\bullet}, \cdots, \mathfrak{u}^h_{n_\bullet+N_\bullet-1}$ are normalized in $\mathfrak{H}$ vector eigenfunctions of the operator $\mathfrak{K}^h$ corresponding to all eigenvalues from segment (119). By formulae (97), and (12), (14),

$$|\langle U^{(k)}_{(p)}, U^{(l)}_{(p)}\rangle - \lambda_p \delta_{k,l}| = o(1) \quad \text{for} \quad h \to +0 .$$

Furthermore, owing to formula (121), we have

$$|\langle U^{(k)}_{(p)}, U^{(l)}_{(p)}\rangle - \lambda_p (\mathfrak{a}^{(k)})^\top \mathfrak{a}^{(l)}| = o(1) \quad \text{for} \quad h \to +0 .$$

Thus, for sufficiently small h the number $N_\bullet$ cannot be smaller than $\kappa^{(q)}_p$. Hence, there are eigenvalues $\mathfrak{m}^h_l, \cdots, \mathfrak{m}^h_{l+\kappa^{(q)}_p-1}$ which verify inequality (120) with the majorant $c^{\alpha_\bullet}_{pq} h^{3+\alpha_\bullet}$ (since the exponent $\alpha \in (0, 1/2)$ is arbitrary, we can choose $\alpha_\bullet < \alpha$ without loosing of the precision in the final estimate (66)). Selecting all eigenvalues of the matrix $\mathcal{M}^{(p)}$, and subsequently the numbers $\lambda_{p-1}, \cdots, \lambda_1$, it turns out that necesserily the equality in (118) occurs, and also $N_\bullet = \kappa^{(q)}_p$.

The proof of Theorem 3.2 is completed. ■

Remark 5.2. *Theorem 3.2 provides inequality* (121)*, which allows for derivation of some asymptotic formulae for vector eigenfunctions* $u^h_{(p)}$ *of the problem* $(12)^h$*. We emphasise that, first, the estimates of remainder are not as good as in the case of eigenvalues, and, second, for multiple eigenvalues of matrix* $\mathcal{M}^{(p)}$ *even the initial approximation for* $u^h_{(p)}$ *is not available. And this is not a lack of the obtained estimates but just the matter of asymptotic procedures; we refer the reader to the chapter 7 of book* [24] *and to papers* [25, 26, 12, 13]*, where is discussed the notion of individual and collective asymptotics of solutions to spectral problems. We present one variant of the estimates proved above.*

If $\mu^{(q)}_p$ *is a simple eigenvalue of the matrix* $\mathcal{M}^{(p)}$ *(for example,* λ_p *is a simple eigenvalue of problem* (12)*) and* $b^{(q)}$ *the corresponding normalized eigenvector, then there is an eigenvalue* λ^h_q *in problem* (12)*) (if* λ_p *is simple than* $p = q$*), which is simple, and together with the corresponding vector eigenfunction verifies the estimates*

$$|\lambda^h_q - \lambda_p - h^3 \mu^{(q)}_p| \le c_p(\alpha) h^{3+\alpha} ,$$

$$\|u^h_{(p)} - (b^{(q)}_1 u_{(p)} + \cdots + b^{(q)}_{\varkappa_p} u_{(p+\varkappa_p-1)}); H^1(\Omega)\| \le C_p(\alpha) h^\alpha ,$$

where $\alpha \in (0, 1/2)$ *is arbitrary, and the factors* $c_p(\alpha)$*,* $C_p(\alpha)$ *are independent of parameter* $h \in (0, h_p]$*.*

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